MODELLING OF CONNECTIONS OF SHELL AND BEAM ELEMENTS IN FINITE ELEMENT ANALYSIS

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SUMMARY

The modelling of the connection of a beam with shell or solid elements in finite element analysis is considered. In the beam the stresses are summed up to stress resultants (longitudinal forces and bending moments) whereas for shell or solid elements only distributed pressures are allowed in order to avoid stress singularities. Different models to represent the beam-shell or beam-solid element connections are currently in use. All of these models possess some inconsistencies. In a widely used model the connection is defined by rigid body kinematics. This corresponds to a linear distribution of the displacements at the beam cross section according to the Bernouilli-Navier hypothesis. It can be shown that this type of connection overestimates not only the stiffness but also the sectional forces in the beam element at the connection.

The paper presents a more consistent approach for modelling the connection between beam and shell elements based on the assumption of a linear distribution of longitudinal stresses instead of displacements over the cross section of the beam. The stress resultants of the beam element are transformed into the nodal forces of the shell elements by a linear relationship. Similarly a transformation relationship for the displacements is formulated. Both relationships allow the transformation of the stiffness matrix of the beam element on a stiffness matrix relating to the nodal points of the shell elements. In this way the longitudinal as well as the bending stiffness of the beam element is taken into account consistently. The model also gives the normal and shear forces as well as the bending moments in the beam, to be used in design. Examples demonstrate the practical value of the approach.

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MODELLING OF CONNECTION OF SHELL AND BEAM ELEMENTS

1: Introduction

In structural mechanics different stress descriptions are used for plates and beams. The most general description is the stress tensor used for three-dimensional solids or plates in plane stress. For beams and plates in bending, stress resultants like bending moments and shear forces are successful engineering concepts. Sometimes it is efficient to use both stress descriptions in the same finite element model. However the basic differences in both models may result in problems. A point force e.g. which is a successful concept in beam models causes stress and displacement singularities in plane stress models.

For the connection of beams with plane stress elements the transition between stress resultants and stresses has to be modelled. In the beam element, the stresses are integrated to stress resultants as longitudinal forces, shear forces and bending moments. For the plate, only distributed loads are allowed in order to avoid stress and displacement singularities. In addition the moment transfer at the end of the beam element has to be modelled, Fig. 1a. Hence, the consistent modelling of the connection of a beam element with plane stress elements is not obvious. Another typical problem is the connection of beams with plates in bending when the beam acts perpendicular to the plate, as encountered in the connection of flat slabs and columns of reinforced concrete buildings.

2: Models for Element Transitions

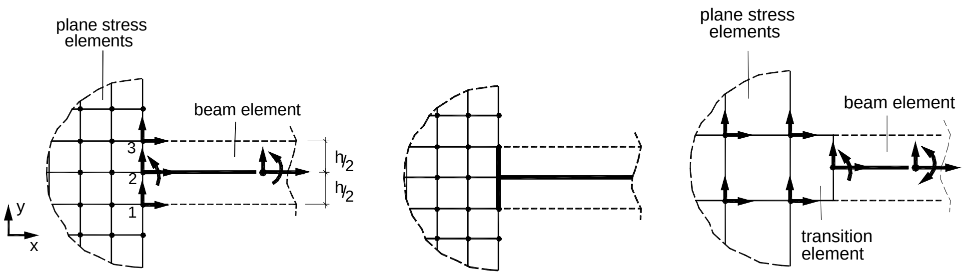
There are various ways to model transitions between different element types. These are

- Multipoint constraints
    - Transformation method
    - Lagrange multiplier method
    - Penalty method
- Transition elements
- Engineering approaches
- Equivalent stiffness transformation (EST)

For multipoint constraints it is assumed that the displacements of the nodes can be described by rigid elements (Kugler 1999). For the beam-plate connection in Fig. 1(a), e.g., these rigid constraints can be described for the nodes 1, 2 and 3 as

$$u_1 = u_2 + \varphi_2 \cdot h/2\,, \qquad u_3 = u_2 - \varphi_2 \cdot h/2\,, \qquad v_1 = v_2\,, \qquad v_3 = v_2\,.$$

The degrees of freedom of the ‘slave’ nodes 1 and 3 are expressed by the translations and the rotation of ‘master’ node 2. This method is called the transformation method. Other methods to take into account multipoint constraints are the Lagrange multiplier method and the Penalty method.



(a) Multipoint constraints (b) Engineering models (c) Transition element

Figure 1: Models for element transitions.

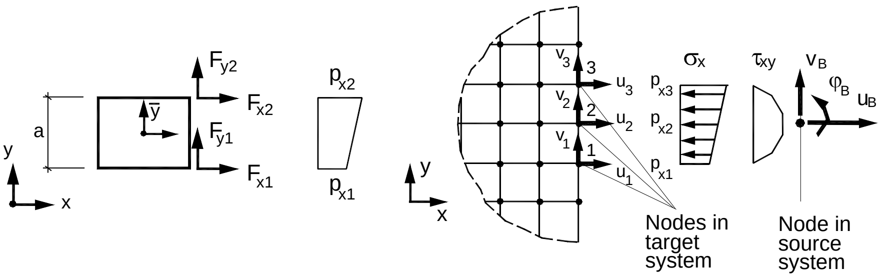
Transition elements are another approach to connect dissimilar finite element domains. They are special finite elements to connect two elements of different types, Fig. 1(c).

For engineering purposes, models as shown in Fig. 1(b) are used. The short beam elements act as rigid constraints. If the stiffness of the beam elements chosen is large enough, the model fulfils approximately the multipoint constraint conditions (Cook 1995).

A new approach for element transitions is developed based on the fulfilment of the equilibrium conditions and the approximate compatibility of the displacements between the two elements. It is called Equivalent Stiffness Transformation or EST. The model avoids rigid constraints, allows multiple element connections, doesn’t result in stress singularities and hence is suited for adaptive meshing.

3: Equivalent Stiffness Transformation (EST)

In the Equivalent Stiffness Transformation two stress systems, the source system and the target system are considered. The stiffness matrix formulated in the source system is transformed by EST into the target system in which the global equations are formulated (Werkle 2001).



(a) Finite Element (b) Source and target system

Figure 2: **EST Model.**

In the example of the beam-plane stress element connection, the source system is the linear stress distribution in the beam which can be integrated to give stress resultants. The stiffness matrix of the beam corresponds to the end forces and moments of the beam and the corresponding degrees of freedom. The target system is the plate in plane stress. Its degrees of freedom are the nodal displacements of the element. Hence the stiffness matrix of the beam element relating to the degrees of freedom u_B, v_B and φ_B is to be transformed into the degrees of freedom $u_1, v_1, u_2, v_2, u_3, v_3$ of the target system, Fig. 2.

Method of transformation

The transformation of the stiffness matrix in the source system is done in three steps (Werkle 2002a):

Step 1: Determination of the stresses of the source system at the nodes of the target system

In the source system the stresses are expressed by the nodal forces $\underline{F}_S$. The stresses $\underline{p}_T$ at the nodal points of the target system are

$$\underline{p}_T = \underline{X} \cdot \underline{F}_S \,. \tag{1}$$

Step 2: Determination of nodal forces in the target system for the stress pattern according to step 1

The stresses are now applied as “distributed loads” to the elements of the target system. The nodal forces corresponding to these element stresses are

$$\underline{F}_T = \underline{A} \cdot \underline{p}_T \tag{2}$$

The Matrix A is obtained by assembling element matrices $\underline{A}^{(el)}$ of all elements connected to the source system. The element assemblage procedure is the same as for element stiffness matrices. The matrix $\underline{A}$ depends on the finite element type in the target system as well as on the function describing the stress variation in the source system.

Step 3: Transformation of the stiffness matrix of the source system

The transformation matrix for the element forces can be obtained easily with (1) and (2) as

$$\underline{F}_T = \underline{T}^T \cdot \underline{F}_S \tag{3}$$

where

$$\underline{T}^T = \underline{A} \cdot \underline{X} \ . \tag{4}$$

The node displacements are $\underline{u}_S$ and $\underline{u}_T$ in the source and in the target system, respectively. It can be shown that they are also transformed with the matrix $\underline{T}$ as

$$\underline{u}_S = \underline{T} \cdot \underline{u}_T \tag{5}$$

In the source system the stiffness matrix is given by

$$\underline{K}_S \cdot \underline{u}_S = \underline{F}_S \tag{6}$$

Using (3) and (5) it can be transformed into the target system by

$$\underline{K}_T \cdot \underline{u}_T = \underline{F}_T \tag{7}$$

with $\underline{K}_T = \underline{T}^T \cdot \underline{K}_S \cdot \underline{T}$. (8)

4: Applications

EST for a Beam and Plane Stress Elements

The application of the procedure described above is done for the connection of a beam element with isoparametric plane stress elements with linear interpolation functions. The beam representing the source system has the degrees of freedom and nodal forces (see Fig. 2b) to be connected with the plate

$$\underline{u}_S = \underline{u}_a = \begin{bmatrix} u_B \\ v_B \\ \varphi_B \end{bmatrix} \qquad \underline{F}_S = \underline{F}_a = \begin{bmatrix} F_{x,B} \\ F_{y,B} \\ M_{z,B} \end{bmatrix}. \tag{9}$$

The stiffness matrix of the beam with the nodes a and b can be written

$$\begin{bmatrix} \underline{K}_{aa} & \underline{K}_{ab} \\ \underline{K}_{ba} & \underline{K}_{bb} \end{bmatrix} \cdot \begin{bmatrix} \underline{u}_a \\ \underline{u}_b \end{bmatrix} = \begin{bmatrix} \underline{F}_a \\ \underline{F}_b \end{bmatrix} \tag{10}$$

If two elements with linear interpolation function are connected with the beam element the transformation matrix eq. (4) is .

$$\underline{T} = \begin{bmatrix} 1/4 & 0 & 1/2 & 0 & 1/4 & 0 \\ 0 & 1/8 & 0 & 3/4 & 0 & 1/8 \\ 1/d & 0 & 0 & 0 & -1/d & 0 \end{bmatrix} \qquad \underline{u}_T = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \quad \text{and} \quad \underline{u}_S = \underline{T} \cdot \underline{u}_T . \tag{11}$$

Eq. (11) illustrates the averaging process of the transformation matrix for the displacements.

The transformation of node a by EST gives the modified stiffness matrix of the beam

$$\begin{bmatrix} \underline{T}^T \cdot \underline{K}_{aa} \cdot \underline{T} & \underline{T}^T \cdot \underline{K}_{ab} \\ \underline{K}_{ba} \cdot \underline{T} & \underline{K}_{bb} \end{bmatrix} \cdot \begin{bmatrix} \underline{u}_T \\ \underline{u}_b \end{bmatrix} = \begin{bmatrix} \underline{F}_T \\ \underline{F}_b \end{bmatrix}. \tag{12}$$

Node b can be transformed similarly, if necessary.

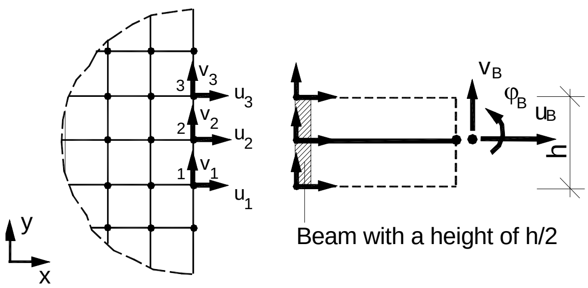


Figure 3: Connection of a beam with plane stress elements in an ESTS model.

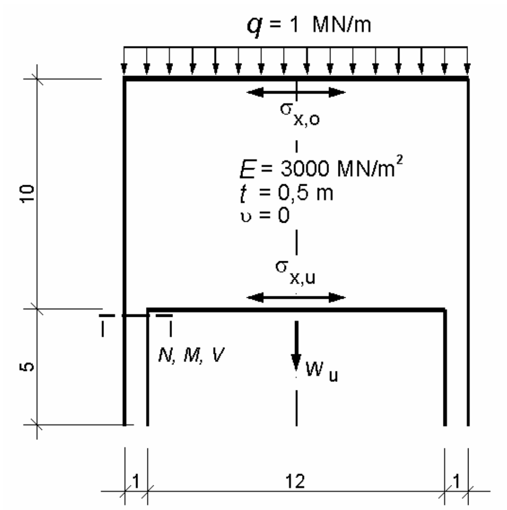
The model neglects the local stiffness of the beam at the connection. It can be taken into account approximately by adding the stiffness of a beam with the height h/2 at the nodes connecting the beam and the shell elements, e.g. nodes 1-2 and 2-3 in Fig. 3. The model with the stiffness extension is called ESTS.

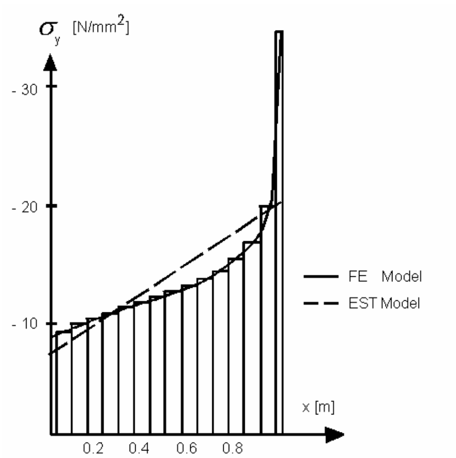
Example: Deep beam with columns

The Equivalent Stiffness Transformation can be applied to any beam-to-shell-connection. The case of a deep beam supported by two columns is considered, Fig. 4 (Werkle 2004). The columns are modelled by

- finite elements (FEM)
- beams with a rigid multipoint constraint connection (MPC)
- beams with a EST connection
- beams with a ESTS connection

The element size is varied. Fig. 5 shows the finite element model for an element size of 0.5 [m] i.e. 2 elements per [m] or n=2. The stresses and the displacements of the FEM model at selected points agree well with the EST model, Tab. 1.



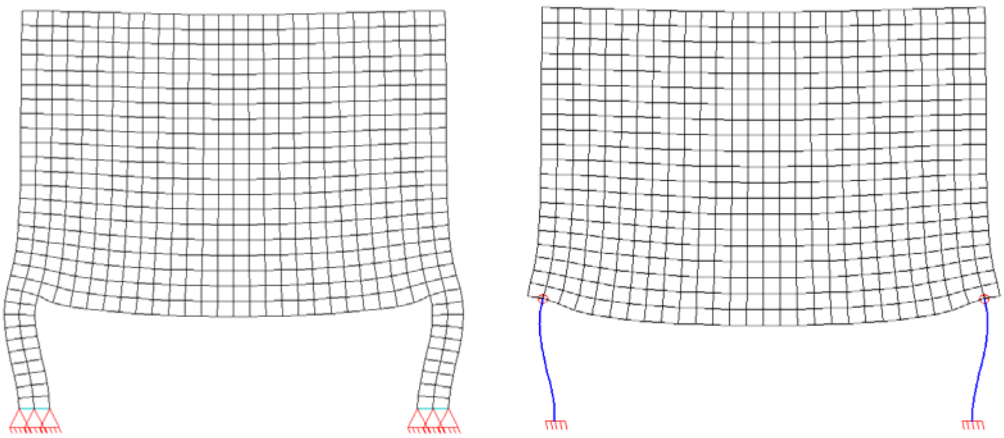


Vertical stresses in section I-I, n=15

Figure 4: Deep beam.

Table 1: Stresses and displacements - FEM versus EST

stresses & displacements		n			
		1	2	4	8
FEM/ EST	$\sigma_{x,o}$ [N/mm²]	–1.80 / -1.83	–1.91 / -192	–1.94 / -195	–1.95/-1.96
	$\sigma_{x,u}$ [N/mm²]	3.51 / 3.59	3.60 / 3.63	3.64 / 3.65	3.65 / 3.66
	w_u [mm]	3.90 / 393	3.99 / 4.00	4.02 / 4.03	4.03 / 4.04



(a) FEM Model (b) EST Model

Figure 5: Deformed Finite Element Models (n=2).

The convergence of the bending moment of the column in section I-I is shown in Fig. 6. It can be seen that the multipoint constraint model behaves too rigid whereas the EST model is too soft. The improved ESTS model converges to the finite element solution. For moderately fine meshing with n=2 to n=4, however, the EST model gives the best results. In this case the slight overestimation of the stiffness of the isoparametric plane stress elements is compensated by the slight underestimation of the stiffness by the pure EST model.

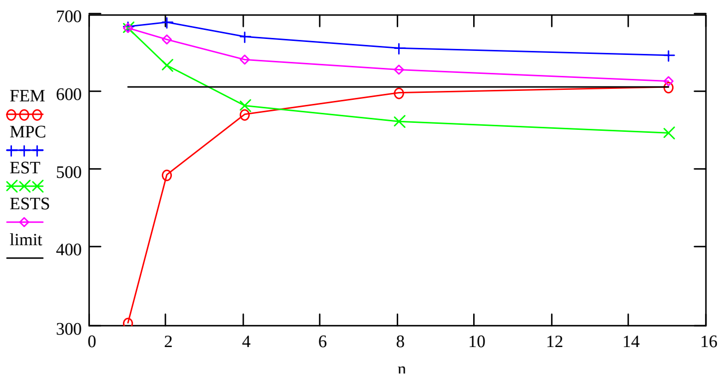


Figure 6: Convergence of the column bending moment [kNm] in section I- I.

EST for Beams acting perpendicular to a plate in bending

Beams perpendicularly acting to a plate are typical for columns on flat slabs in reinforced concrete constructions, Fig. 7.

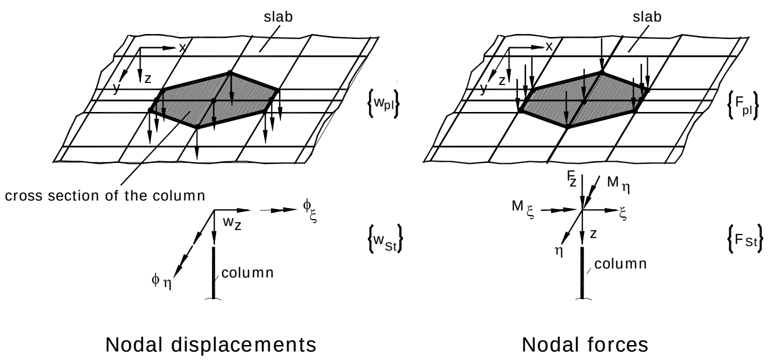


Figure 7: Connection of a Column with a Reinforced Concrete Slab.

Assuming again a linear distribution of the longitudinal stresses according to the Bernouilli-Navier hypothesis the procedure described above can be applied. The transformation matrix is given for the case of a beam with rectangular cross section and a discretisation of the plate in four finite elements in the beam section acc. to Fig. 8 (Werkle 2000).

With the displacements $\underline{w}_T$ of the nodal points perpendicular to the plate and the rotational and translational degrees of freedom of the beam $\underline{w}_S$ as

$$\underline{w}_T = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{bmatrix}, \qquad \underline{w}_S = \begin{bmatrix} w_z \\ \phi_{yy} \\ \phi_{xx} \end{bmatrix} \tag{13}$$

respectively, the transformation matrix according to Eq. (8) is obtained as

$$\underline{T} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{4 \cdot d_x} & 0 & \frac{-1}{4 \cdot d_x} & \frac{1}{2 \cdot d_x} & 0 & \frac{-1}{2 \cdot d_x} & \frac{1}{4 \cdot d_x} & 0 & \frac{-1}{4 \cdot d_x} \\ \frac{1}{4 \cdot d_y} & \frac{1}{2 \cdot d_y} & \frac{1}{4 \cdot d_y} & 0 & 0 & 0 & \frac{-1}{4 \cdot d_y} & \frac{-1}{2 \cdot d_y} & \frac{-1}{4 \cdot d_y} \end{bmatrix} \tag{14}$$

With Eq. (4) the displacements of the source system (beam) can be understood as a weighted average of the displacements of the target system (plate).

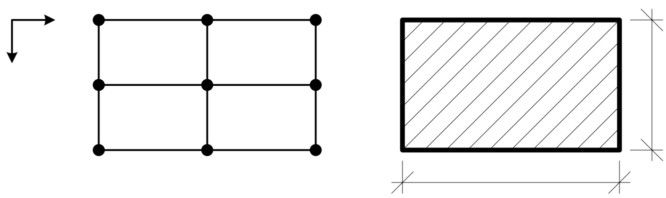
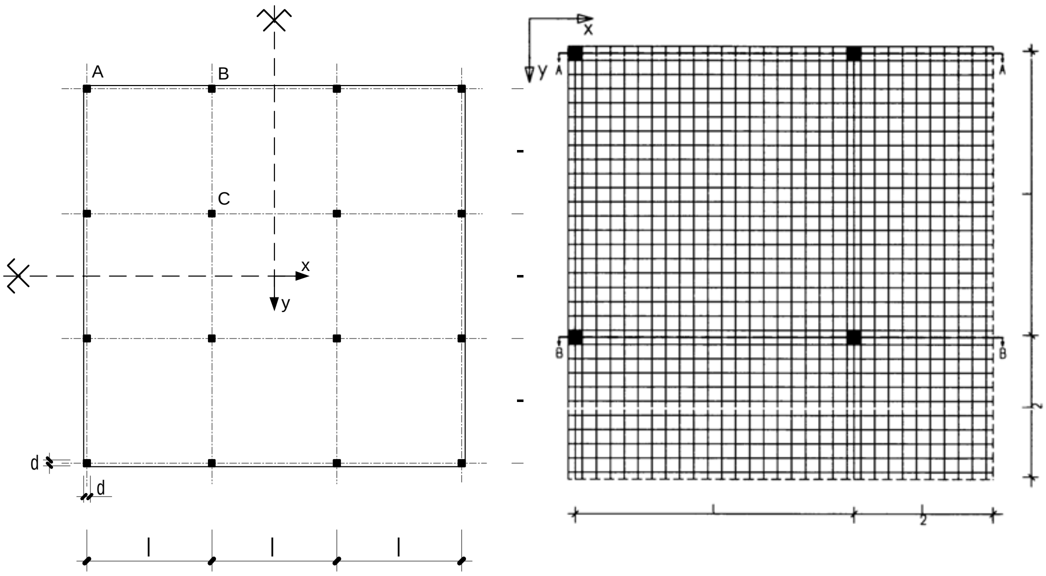


Figure 8: Finite element assemblage for a rectangular column.

Example: Regular flat slab with 3x3 panels

The slab is loaded by a constant distributed load p, its thickness is $d_{Pl} = \ell/30$, the Poisson ratio is $\mu = 0.2$, Fig. 9 (Werkle 2002b). The quadratic columns with the dimensions $a = b = \ell/20$ and the height $h_S = \ell/2$ are pin-supported at the lower end.



(a) Flat with 3x3 panels

(b) FE model of a quarter of the slab

Figure 9: Finite element model of a flat slab.

For comparison two models with an elastic support by Winkler springs are investigated. The Winkler moduli have been chosen as $k_{s\_z} = E/h_S$ corresponding to the normal stiffness of the column as well as $k_{s\_z} = 3 \cdot E/h_S$ corresponding to its bending stiffness.

Figure 10 shows the bending moment m_x in section B-B for the two spring constants and for the EST element. In the corner column and in the edge column, the column stiffness influences the bending moments considerably. However, at the internal panel its influence can be neglected. The results of the elastic support with a Winkler modulus of $k_{s\_z} = 3 \cdot E/h_S$ agree well with the EST model. The stress resultants in the column are given in Table 2.

Table 2: Stress Resultants in the columns (beam)

Column	$F_z/(p\cdot\ell^2)$	$M_x/(p\cdot\ell^3)$	$M_y/(p\cdot\ell^3)$
A	0.219	0.0134	-0.0134
B	0.474	0.0218	0.0003
C	1.157	-0.0004	0.0004

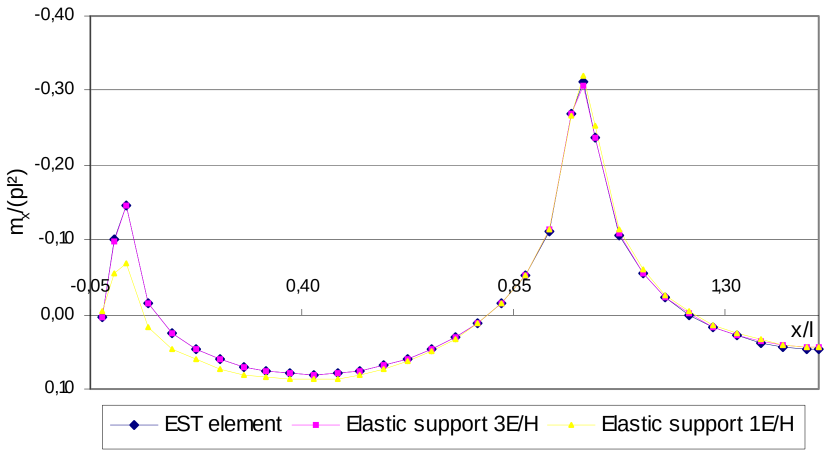


Figure 10: Bending moment m_x, section B-B.

It should be mentioned that the EST method can also be applied to any other type of beam cross section and to highly sophisticated structures as e.g. the modelling of complete buildings, Fig. 11.

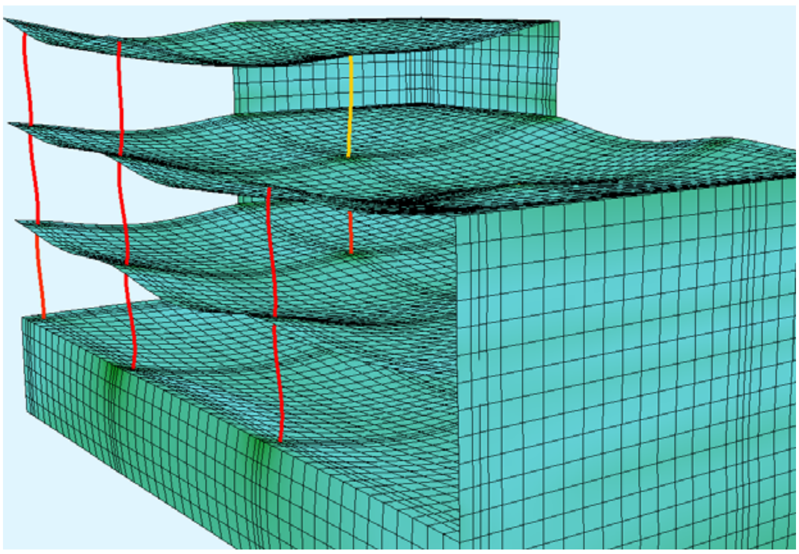


Figure 11: Model of a RC building (Gerold, 2004).

5: Conclusions

The EST is well suited to model the connection between domains of finite elements with different stress systems. The results agree well with more sophisticated finite element models. EST can be applied to connect structural elements with different stress systems as e.g. beams with plate elements. In EST-beams, the sectional forces which are required in design are obtained directly without numerical integration of element stresses. The model can be extended to any other type of beam to solid connection.

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