Structural Mechanics Problems with Uncertain Node Locations

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Outline

- Interval arithmetic
- Parametric systems of linear equations
- Statically determinate example and Method 1 (interval tightening)
- Statically indeterminate example and Method 2 (monotonicity analysis)
- Conclusions

Interval arithmetic

Let $\mathbb{IR}$ denote the set of the compact, nonempty real intervals. The arithmetic operation $\circ \in \{+, -, \cdot, /\}$ on $\mathbb{IR}$ is defined in the following way. If $a = [\underline{a}, \overline{a}], b = [\underline{b}, \overline{b}] \in \mathbb{IR}$, then

$$\begin{aligned}
a + b &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
a - b &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
a \cdot b &= [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}], \\
a \,/\, b &= [\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}, \max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}], \text{ if } 0 \notin b.
\end{aligned}$$

Interval arithmetic (dependency problem)

Note that some relations known to be true in the set $\mathbb{R}$, e.g. the distributive law, are not valid in $\mathbb{IR}$. Here we have the weaker subdistributive law

$$a \cdot (b + c) \subseteq ab + ac \text{ for } a, b, c \in \mathbb{IR}.$$

Example:

$a := [-1, 1]$, $b := [0, 2]$, $c := [-3, 1]$

$a \cdot (b + c) = [-1, 1] \cdot ([0, 2] + [-3, 1]) = [-1, 1] \cdot [-3, 3] = [-3, 3]$

$ab + ac = [-1, 1] \cdot [0, 2] + [-1, 1] \cdot [-3, 1] = [-2, 2] + [-3, 3] = [-5, 5]$

Parametric Linear Systems

Given: $A(p) \cdot x = b(p)$,

where $A(p) \in \mathbb{R}^{n\times n}$, $b(p) \in \mathbb{R}^n$ are depending on $p \in [p] = ([p_1], \ldots, [p_k])^T$.

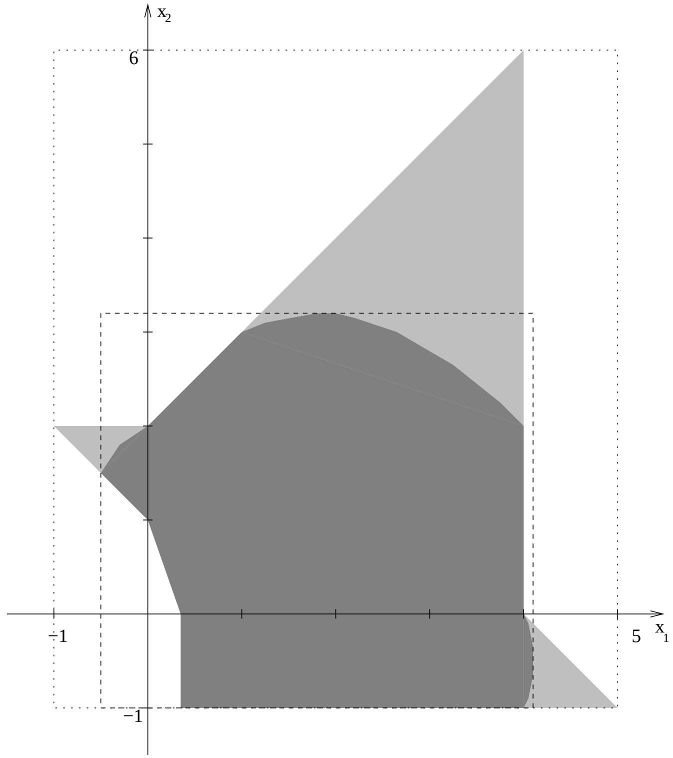
Parametric solution set

$$\Sigma_p = \Sigma(A(p), b(p), [p]) := \{x \in \mathbb{R}^n \,|\, A(p) \cdot x = b(p) \text{ for some } p \in [p]\}$$

wanted: $\square\Sigma_p = [\inf \Sigma_p, \sup \Sigma_p]$ or a tight enclosure for it.

The solution set of parametric linear systems

$$\begin{pmatrix} p_1 & p_2 \\ p_2 & -p_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} p_3 \\ p_4 \end{pmatrix} \quad \text{with} \quad \begin{array}{c} p_1 \in [1,3],\ p_2 \in [0,1], \\ p_3 \in [1,4],\ p_4 \in [-2,1] \end{array}$$



Approaches for the interval FEM

- Element-by-element formulation with a penalty method (R. L. Muhanna, R. L. Mullen, ...)

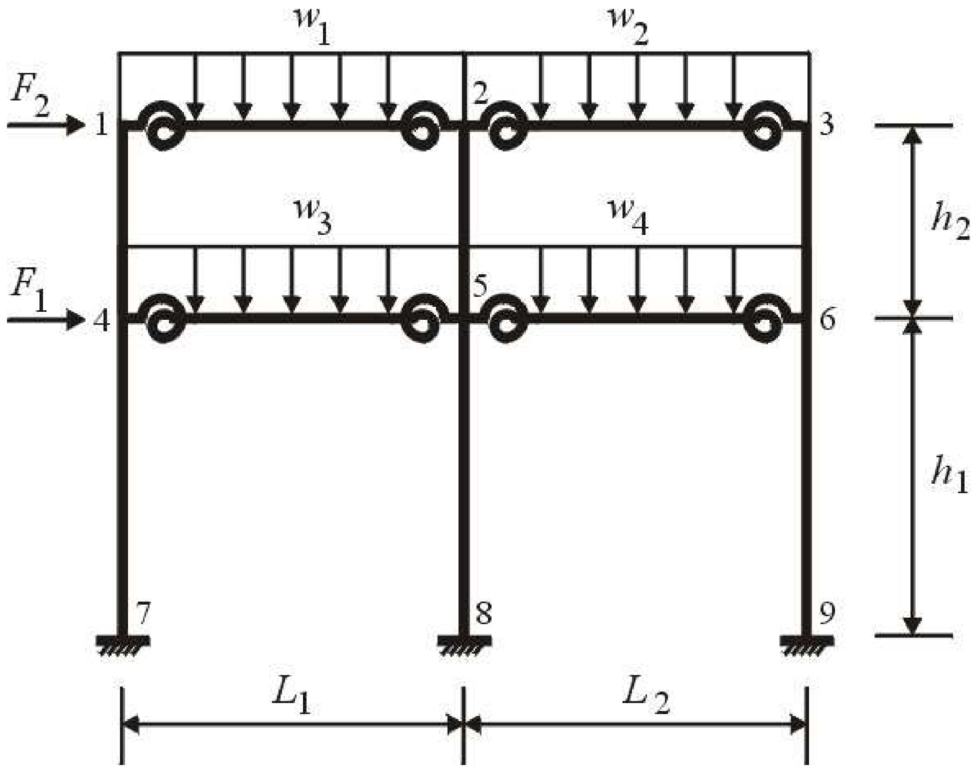
- Bounding interval system solutions using centred forms (A. Neumaier, A. Pownuk)

- Solution of parametric systems by parametric residual iteration (E. D. Popova, ...)

Uncertain Material Properties and Loadings

Example: Two-Bay Two-Story Frame

(E. D. Popova, R. Iankov, Z. Bonev)



Example: Parametric linear system of order 18 with 13 uncertain parameters

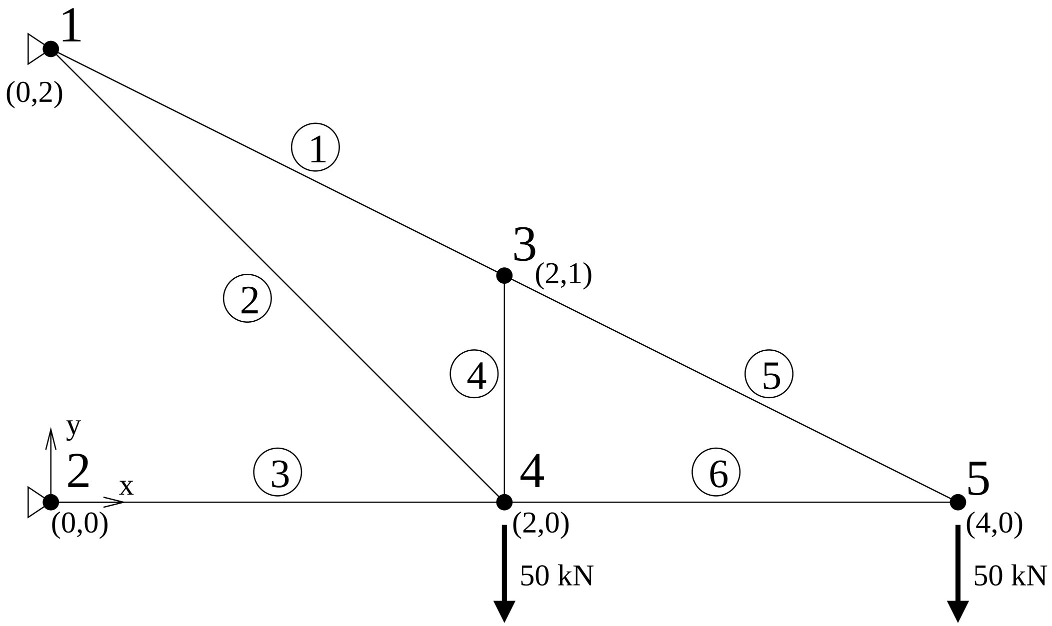
	Columns (HE 280 B)	Beams (IPE 400)
Cross-sectional area	$A_c = 0.01314\,\text{m}^2$,	$A_b = 0.008446\,\text{m}^2$
Moment of inertia	$I_c = 19270 * 10^{-8}\,\text{m}^4$,	$I_b = 23130 * 10^{-8}\,\text{m}^4$
Modulus of elasticity	$E_c = 2.1 * 10^8\,\text{kN/m}^2$,	$E_b = 2.1 * 10^8\,\text{kN/m}^2$
Rotational spring stiffness	$c = 10^8\,\text{kN}$	
Uniform vertical load	$w_1 = \ldots = w_4 = 30\,\text{kN/m}$	
Concentrated lateral forces	$f_1 = f_2 = 100\,\text{kN}$	

tolerances: material properties 1%, spring stiffness and all applied loadings 10%

time: ca. 1.3 s

Uncertain Material Properties, Loadings, and Node Locations

Statically determinate example: mechanical truss model with six elements



Interval parameters for the six-element truss model

Parameter		Nominal Value	Uncertainty
Young's m. * area	EA	422100 kN	±21105 kN (±5%)
	(x_1, y_1)	$(0, 2)$	±0.005 m
	(x_2, y_2)	$(0, 0)$	±0.005 m
Node coordinates	(x_3, y_3)	$(2, 1)$	±0.005 m
	(x_4, y_4)	$(2, 0)$	±0.005 m
	(x_5, y_5)	$(4, 0)$	±0.005 m
	F_{x_3}, F_{y_3}	0 kN, 0 kN	±1 kN
Loading forces	F_{x_4}, F_{y_4}	0 kN, −50 kN	±1 kN
	F_{x_5}, F_{y_5}	0 kN, −50 kN	±1 kN

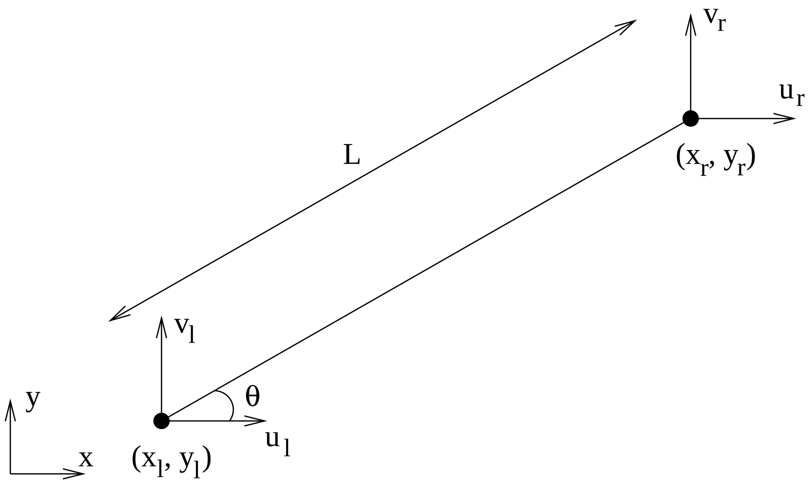
Methodology (Method 1)

Our solution procedure consists of the following stages:

1. Application of a variable substitution to generate the symbolic system stiffness matrix appearing in the FEM in terms of the interval parameters.
2. Initial enclosures for the node displacements obtained by applying a parametric solver (method of E. D. Popova) to the interval system.
3. Initial enclosures for the element forces computed from these node displacements.
4. An interval tightening method applied to the element forces.
5. An interval tightening method applied to the node displacements.

Finite element method

Assembly of the element stiffness matrix



$$\cos\theta = \frac{x_r - x_l}{L}$$

$$\sin\theta = \frac{y_r - y_l}{L}$$

$$L = \sqrt{(x_r - x_l)^2 + (y_r - y_l)^2}$$

Element stiffness matrix

$$k = \frac{EA}{L}\begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

$$= \frac{EA}{((x_r-x_l)^2+(y_r-y_l)^2)^{\frac{3}{2}}}.$$

$$\begin{pmatrix} (x_r-x_l)^2 & (x_r-x_l)(y_r-y_l) & -(x_r-x_l)^2 & -(x_r-x_l)(y_r-y_l) \\ (x_r-x_l)(y_r-y_l) & (y_r-y_l)^2 & -(x_r-x_l)(y_r-y_l) & -(y_r-y_l)^2 \\ -(x_r-x_l)^2 & -(x_r-x_l)(y_r-y_l) & (x_r-x_l)^2 & (x_r-x_l)(y_r-y_l) \\ -(x_r-x_l)(y_r-y_l) & -(y_r-y_l)^2 & (x_r-x_l)(y_r-y_l) & (y_r-y_l)^2 \end{pmatrix}$$

Parametric system solution

We now have a system of linear equations for the node displacements $u_i, v_i,\ i = 3, 4, 5$:

$$Ku \ = \ F,$$

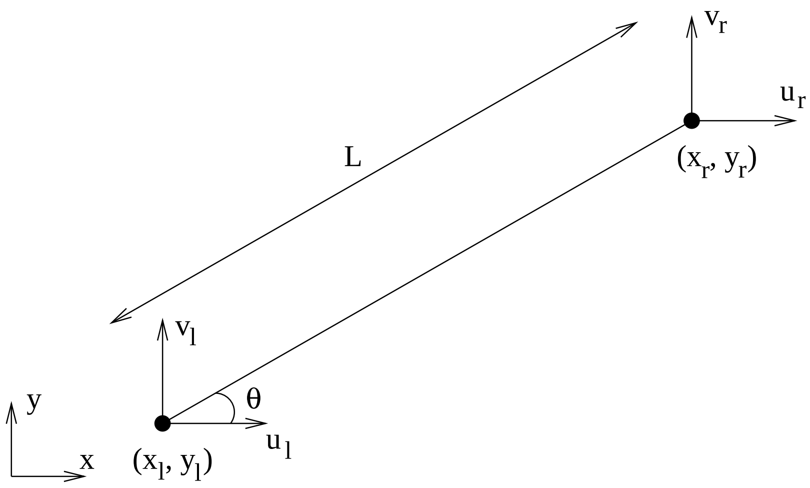
where K is the global system stiffness matrix assembled from the element stiffness matrices k_i, $i = 1, \ldots, 6$, $u = (u_3\ v_3\ u_4\ v_4\ u_5\ v_5)^T$ is the vector of node displacements and $F = (F_{x_3}\ F_{y_3}\ F_{x_4}\ F_{y_4}\ F_{x_5}\ F_{y_5})^T$ is the vector of loading forces.

Initial intervals for the node displacements

The parametric solver computes in about 14.2s (on a PC with an AMD Athlon-64 3GHz processor running the *Mathematica* environment) the following initial node displacement intervals

Node Displacement	**Value**
u_3	$[-0.00064, 0.00058]$
v_3	$[-0.0020, -0.00076]$
u_4	$[-0.0012, -0.00029]$
v_4	$[-0.0021, -0.00068]$
u_5	$[-0.0018, -0.00065]$
v_5	$[-0.0077, -0.0022]$

Element forces



Resulting normal force in element i:

$$
\begin{aligned}
S_i &= \frac{EA}{L}(-\cos\theta \ -\sin\theta \ \cos\theta \ \sin\theta)u_i \\
&= \frac{EA}{(x_r - x_l)^2 + (y_r - y_l)^2}(x_l - x_r \ \ y_l - y_r \ \ x_r - x_l \ \ y_r - y_l)u_i
\end{aligned}
$$

where

$$u_i = (u_l \; v_l \; u_r \; v_r)^T$$

is the vector of displacements for its left- and right-hand nodes.

Initial intervals for the normal forces

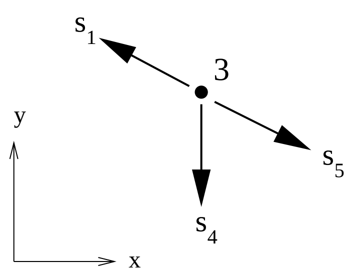
Element Force	Value
S_1	$[-46.0, 283.0]$
S_2	$[-48.9, 198.5]$
S_3	$[-250.3, -57.4]$
S_4	$[-590.7, 589.8]$
S_5	$[-388.5, 622.9]$
S_6	$[-324.9, 113.1]$

Interval tightening

Contraction of the resulting intervals, e.g. for the element forces:
At the free-moving nodes 3, 4, and 5, all forces (element and loading forces) must be in equilibrium, in both the x- and y-directions.
For example, at node 3 we must have

$$\begin{aligned} S_1 \cos\theta_1 + F_{x_3} &= S_4 \cos\theta_4 + S_5 \cos\theta_5 \\ S_1 \sin\theta_1 + F_{y_3} &= S_4 \sin\theta_4 + S_5 \sin\theta_5 \end{aligned}$$

Rearrangement gives

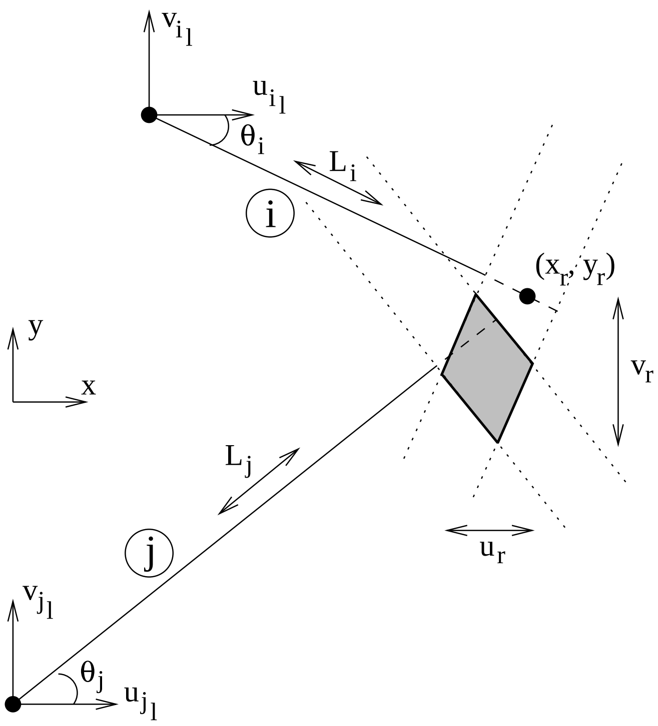


$$\begin{aligned} S_1 &= \frac{S_4 \cos\theta_4 + S_5 \cos\theta_5 - F_{x_3}}{\cos\theta_1} \\ S_1 &= \frac{S_4 \sin\theta_4 + S_5 \sin\theta_5 - F_{y_3}}{\sin\theta_1} \\ &\ldots \end{aligned}$$

Results of interval tightening on the element forces

Element Force	Starting Value
S_1	$[-46.0, 283.0]$
S_2	$[-48.9, 198.5]$
S_3	$[-250.3, -57.4]$
S_4	$[-590.7, 589.8]$
S_5	$[-388.5, 622.9]$
S_6	$[-324.9, 113.1]$
	Iteration 10
S_1	$[105.2, 118.6]$
S_2	$[56.7, 84.8]$
S_3	$[-167.0, -133.3]$
S_4	$[-12.7, 7.4]$
S_5	$[107.0, 116.7]$
S_6	$[-105.7, -94.4]$

Contraction of the node displacements



$$S_i = EA(L_i - L_{i_0})/L_{i_0}$$

Taking each node in turn, the method thus proceeds as follows:

- Take every possible pair of non-parallel elements which meet at the node, in turn.
- For each such pair, compute an interval enclosure for the displacement of their common node, in the x- and y-directions.
- Take the intersections of these two new intervals with the current values for the displacement of the node.

Again, this procedure is iterated (for all nodes) as desired until the resulting set of displacement intervals do not contract any further.

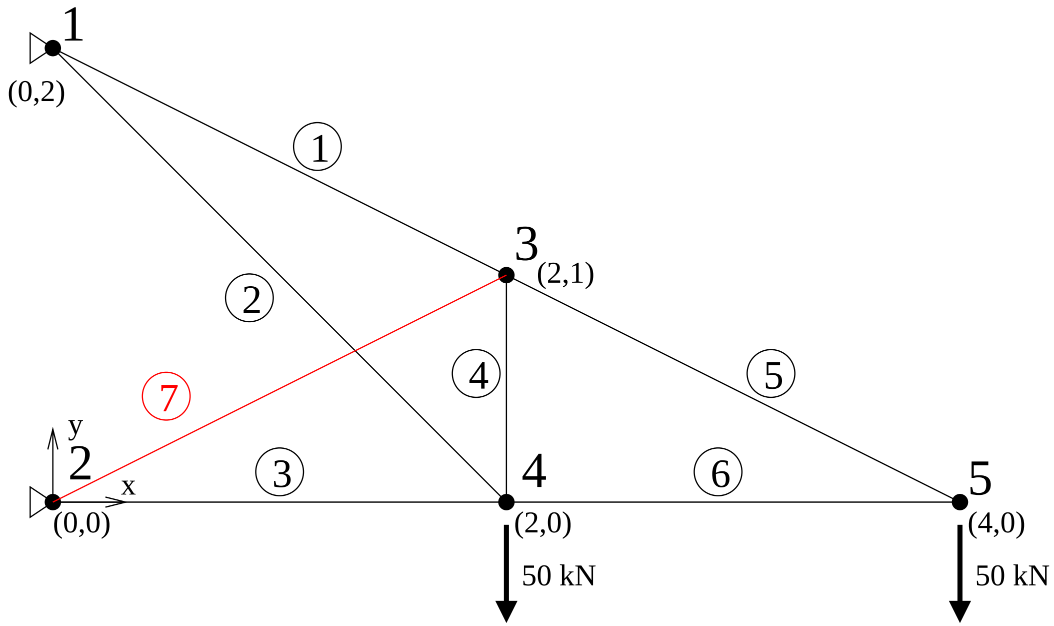
Results of interval tightening on the node displacements

Node Displacement	Starting Value
u_3	$[-0.00064, 0.00058]$
v_3	$[-0.0020, -0.00076]$
u_4	$[-0.0012, -0.00029]$
v_4	$[-0.0021, -0.00068]$
u_5	$[-0.0018, -0.00065]$
v_5	$[-0.0077, -0.0022]$
	Iteration 5
u_3	$[-0.00031, 0.00024]$
v_3	$[-0.0018, -0.0010]$
u_4	$[-0.00087, -0.00057]$
v_4	$[-0.0018, -0.0010]$
u_5	$[-0.0015, -0.00098]$
v_5	$[-0.0059, -0.0042]$

Results

	Inner Estimation (Monte Carlo)	**Outer Estimation** (Param. Sol. & Tight.)	**OE/IE** lwr bd; upr bd	**OE/IE** width
S_1	$[106.8, 116.6]$	$[105.2, 118.6]$	0.98;1.02	1.38
S_2	$[65.2, 76.2]$	$[56.7, 84.8]$	0.87;1.11	2.52
S_3	$[-156.7, -143.8]$	$[-167.0, -133.3]$	1.06;0.93	2.58
S_4	$[-7.7, 2.5]$	$[-12.7, 7.4]$	1.63;2.87	1.94
S_5	$[108.4, 115.3]$	$[107.0, 116.7]$	0.99;1.01	1.35
S_6	$[-104.2, -96.0]$	$[-105.7, -94.4]$	1.01;0.99	1.33
u_3	$[-0.000083, 0.000025]$	$[-0.00031, 0.00024]$	3.69;9.11	4.96
v_3	$[-0.0015, -0.0013]$	$[-0.0018, -0.0010]$	1.16;0.83	2.55
u_4	$[-0.00078, -0.00065]$	$[-0.00087, -0.00057]$	1.10;0.90	2.06
v_4	$[-0.0015, -0.0013]$	$[-0.0018, -0.0010]$	1.14;0.85	2.39
u_5	$[-0.0013, -0.0011]$	$[-0.0015, -0.00098]$	1.08;0.91	1.91
v_5	$[-0.0054, -0.0046]$	$[-0.0059, -0.0042]$	1.07;0.94	1.77

Statically indeterminate example: mechanical truss model with seven elements



Interval parameters for the seven-element truss model

Parameter		**Nominal Value**	**Uncertainty**
Young's m. * area	EA	422100 kN	$\pm$21105 kN ($\pm$5%)
	(x_1, y_1)	$(0, 2)$	$\pm$0.005 m
	(x_2, y_2)	$(0, 0)$	$\pm$0.005 m
Node coordinates	(x_3, y_3)	$(2, 1)$	$\pm$0.005 m
	(x_4, y_4)	$(2, 0)$	$\pm$0.005 m
	(x_5, y_5)	$(4, 0)$	$\pm$0.005 m
	F_{x_3}, F_{y_3}	0 kN, 0 kN	$\pm$0 kN
Loading forces	F_{x_4}, F_{x_5}	0 kN, 0 kN	$\pm$0 kN
	F_{y_4}, F_{y_5}	-50 kN, -50 kN	$\pm$1 kN

Monotonicity analysis

We have the system of linear equations for the node displacements

$$Ku = F.$$

Taking the partial derivatives with respect to a parameter p yields

$$K\frac{\partial u}{\partial p} = \left(\frac{\partial F}{\partial p} - \frac{\partial K}{\partial p}u\right),$$

where $p \in \{u_i, v_i \mid i = 3, 4, 5\}$.

Methodology (Method 2)

The new solution procedure consists of the following stages (an interval system solver is used throughout):

1. Construct the system of interval equations $Ku = F$. The widths of the intervals appearing in the element stiffness matrices can be minimised by elementary analysis.
2. Solve to yield an initial enclosure $u^{\{0\}}$ for the node displacements.
3. Construct systems of interval equations for the partial derivatives of the node displacements with respect to each interval parameter, using the current enclosure $u^{\{i\}}$. Solve to yield outer enclosures for the partial derivatives; where zero is excluded, monotonicity is proven.
4. Attempt to minimise/maximise each solution component in turn by restricting the parameter domain for monotone parameters and thusly reconstructing and solving the original system.
5. Iterate 3–4, using both successively tighter solution enclosures and monotonicity information obtained so far, until as many of the solution components as possible are found to be monotone over the restricted parameter domains.

Results of monotonicity analysis on the node displacements

time: ca. 1 s

Node Displacement	**Starting Value**
u_3	$[0.00018, 0.00056]$
v_3	$[-0.0019, -0.00058]$
u_4	$[-0.0012, -0.00037]$
v_4	$[-0.0020, -0.00063]$
u_5	$[-0.0020, -0.00071]$
v_5	$[-0.0087, -0.0036]$
	Iteration 2
u_3	$[0.00028, 0.00039]$
v_3	$[-0.0019, -0.00059]$
u_4	$[-0.00098, -0.00039]$
v_4	$[-0.0012, -0.00082]$
u_5	$[-0.0016, -0.00082]$
v_5	$[-0.0058, -0.0045]$

Monotonicity Summary

Parameter:	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	F_{y_4}	F_{y_5}	EA
u_3	$-$		$(+)$	$(-)$	$-$	$(-)$		$+$		$+$		$-$	$-$
v_3	$+$	$+$	$+$	$-$							$+$	$+$	$+$
u_4			$+$	$-$		$(+)$			$-$		$+$	$+$	$+$
v_4	$+$	$+$	$+$	$-$		$+$		$-$	$-$	$-$	$+$	$+$	$+$
u_5			$+$	$-$	$+$		$-$				$+$	$+$	$+$
v_5	$+$	$(+)$		$-$	$+$	$+$	$-$		$-$	$+$	$+$	$+$	$+$

Results

	Inner Estimation (Monte Carlo)	**Outer Estimation** (Param. Sol. & Tight.)	**OE/IE** lwr bd; upr bd	**OE/IE** width
u_3	$[0.00031, 0.00036]$	$[0.00028, 0.00039]$	0.93;1.07	1.82
v_3	$[-0.0010, -0.0007]$	$[-0.0019, -0.00059]$	1.76;0.68	7.38
u_4	$[-0.00066, -0.00056]$	$[-0.00098, -0.00039]$	1.46;0.71	5.29
v_4	$[-0.0010, -0.00092]$	$[-0.0012, -0.00082]$	1.03;0.90	1.68
u_5	$[-0.0011, -0.00099]$	$[-0.0016, -0.00082]$	1.26;0.83	3.25
v_5	$[-0.0055, -0.0047]$	$[-0.0058, -0.0045]$	1.04;0.97	1.37

Conclusions

- The dependency problem in interval arithmetic necessitates special solution procedures for problems with interval parameters
- Uncertain node locations make the problem more difficult than uncertain material values and loadings alone
- Parametric and interval system solvers can yield rough initial solutions
- Interval tightening procedures and monotonicity analysis can improve these result intervals
- Future work: explore applicability to larger systems